C*-ALGEBRAS ASSOCIATED NONCOMMUTATIVE CIRCLE AND THEIR K-THEORY
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ABSTRACT. In this article we investigate the universal C*-algebras associated to certain 1- dimensional simplicial flag complexes which describe the noncommutative circle. We denote it by $S_{nc}^1$. We examine the K-theory of this algebra and the subalgebras $S_{nc}^1/I_k, I_k$. Where $I_k$, for each $k$, is the ideal in $S_{nc}^1$ generated by all products of generators $h_s$ containing at least $k + 1$ pairwise different generators. Moreover we prove that such algebra divided by the ideal $I_2$ is commutative.

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1. Introduction

J. Cuntz in [2] associate to every simplicial complex a universal $C^*$-algebra with generators and relations. In the following we give some examples and properties of such algebras see also [6].

Definition 1.1. A simplicial complex $\Sigma$ consists of a set of vertices $V_\Sigma$ and a set of non-empty subsets of $V_\Sigma$, the simplexes in $\Sigma$, such that:

- If $s \in V_\Sigma$, then $\{s\} \in \Sigma$.
- If $F \in \Sigma$ and $0 \neq E \subset F$ then $E \in \Sigma$.

$\Sigma$ is called locally finite if every vertex of $\Sigma$ is contained in only finitely many simplexes of $\Sigma$, and finite-dimensional (of dimension $\leq n$) if it contains no simplexes with more than $n + 1$-vertices.

For a simplicial complex $\Sigma$ one can define the topological space $|\Sigma|$ associated to this complex. It is called the "geometric realization" of the complex and can be defined as the space of maps $f : V_\Sigma \rightarrow [0, 1]$ such that $\sum_{s \in V_\Sigma} f(s) = 1$ and $f(s_0), \ldots, f(s_i) = 0$ whenever $\{s_0, \ldots, s_i\} \notin \Sigma$.

If $\Sigma$ is locally finite, then $|\Sigma|$ is locally compact.

- $C_\Sigma$ is the universal $C^*$-algebra with positive generators $h_s, s \in V_\Sigma$, satisfying the relations $h_{s_0} h_{s_1} \ldots h_{s_n} = 0$ whenever $\{s_0, s_1, \ldots, s_n\} \notin \Sigma$.

  \[ \sum_{s \in V_\Sigma} h_s h_t = h_t \quad \forall t \in V_\Sigma. \]

  Here the sum is finite, because $\Sigma$ is locally finite.

- $C_{\Sigma}^{ab}$ is the abelian version of the universal $C^*$-algebra above, i.e. satisfying in addition $h_s h_t = h_t h_s$ for all $s, t \in V_\Sigma$.

Remark 1.1. There exists a canonical surjective map $C_\Sigma \rightarrow C_{\Sigma}^{ab}$.

A simplicial map between two simplicial complexes $\Sigma$ and $\Sigma'$ is a map $\varphi : V_\Sigma \rightarrow V_{\Sigma'}$ such that, whenever $(t_0, \ldots, t_n)$ is a simplex in $\Sigma$ this implies that $(\varphi(t_0), \ldots, \varphi(t_n))$ is a simplex in $\Sigma'$.

Proposition 1.1. Every simplicial map $\varphi : \Sigma \rightarrow \Sigma'$ between two simplicial complexes $\Sigma$ and $\Sigma'$ induces a $*$-homomorphism $\varphi^* : C_{\Sigma'} \rightarrow C_\Sigma$.

Proof. Define $\varphi^* : C_{\Sigma'} \rightarrow C_\Sigma$ by $h_{s_0} \mapsto g_s := \sum_{t} \varphi(t)=s h_t$ and $h_s$ mapped to 0 if $s$ is not in the image of $\varphi$. We verify that the sum of all $g_s$ over $s$ is equal to if one the sum of all $h_t$ over $t$ is equal to one and the products $g_{s_0} \ldots g_{s_n} = 0$ whenever $h_{t_0} \ldots h_{t_n} = 0$.

For the first condition, we have

\[ \sum_{s} g_{s} = \sum_{s}(\sum_{t} \varphi(t)=s h_{t}) = \sum_{t} h_{t} = 1, \]

and for the second condition

\[ g_{s_0} \ldots g_{s_n} = \sum_{\varphi(t_0)=s_0} \ldots \sum_{\varphi(t_n)=s_n} h_{t_0} \ldots h_{t_n} = \sum_{\varphi(t_0)=s_0} \ldots \sum_{\varphi(t_n)=s_n} h_{t_0} h_{t_0} \ldots h_{t_n} = 0 \]

because $\varphi$ is a simplicial map.

It has been shown in [2] that the $K$-theory of $C_\Sigma$ coincides with the $K$-theory of $C_{\Sigma}^{ab}$ (which in turn is isomorphic to $C_0(|\Sigma|)$). In the sequel we will study the $K$-theory of another $C^*$-algebra that can be associated with certain complexes.
Definition 1.2. A simplicial complex $\Sigma$ is called flag or full, if it is determined by its 1-simplices in the sense that 
\[ \{s_0, \ldots, s_n\} \in \Sigma \iff \{s_i, s_j\} \in \Sigma \text{ for all } 0 \leq i < j \leq n. \]

Definition 1.3. Let $\Sigma$ be a locally finite flag complex. Denote by $V$ the set of its vertices. Define $C_{\Sigma}^{flag}$ as the universal $C^*$-algebra with positive generators $h_s, s \in V$, satisfying the relations 
\[ \sum_{s \in V} h_s h_t = h_t, \quad t \in V \]
and 
\[ h_s h_t = 0 \quad \text{for} \quad \{s, t\} \notin \Sigma. \]
Denote by $I_k$ the ideal in $C_{\Sigma}^{flag}$ generated by products containing at least $n + 1$ different generators. The filtration $(I_k)$ of $C_{\Sigma}^{flag}$ is called the skeleton filtration.

For simplicity we denote $C_{\Sigma}^{flag}$ by $C_{\Sigma}^{f}$. This algebra is an interesting example of a noncommutative $C^*$-algebra described by a simplicial complex. If we consider the flag complex $\Sigma_{S^1}$ with vertices $\{0^-, 0^+, 1^-, 1^+\}$ and the condition that exactly the edges $\{i^-, i^+\}$ do not belong to $\Sigma_{S^1}$, the geometric realization of $\Sigma_{S^1}$ is the noncommutative circle $S^1$. We consider the universal $C^*$-algebra with 4 positive generators $h_i, i \in V_{S^1} := \{0^-, 0^+, 1^-, 1^+\}$ and satisfying the relations 
\[ \sum_i h_{i^+} + \sum_i h_{i^-} = 1, h_{i^+} h_{i^-} = 0 \quad \forall i \in \{0, 1\}. \]

The algebra described above is exactly the algebra $C_{\Sigma_{S^1}}^{f}$. We will denote it by $S_1^{nc}$. The abelianization of this $C^*$-algebra is isomorphic to the algebra of continuous functions on the circle $S^1$ as shown in [2]. The $K$-theory of $S_1^{nc}$ is described by the following theorem.

Theorem 1.2. [2] The evaluation map $ev : S_1^{nc} \longrightarrow \mathbb{C}$ at the vertex $1^+$, which maps the generator $h_{1^+}$ to 1 and all the other generators to 0, induces an isomorphism in $K$-theory. (The same is true for the evaluation maps, corresponding to the other vertices.)

Let 
\[ \Delta := \{(t_0, \ldots, t_n) \in \mathbb{R}^{n+1} \mid 0 \leq t_i \leq 1, \sum_{i=1}^{n} t_i = 1\} \]
be the standard $n$-simplex. Denote by $C_{\Delta}$ the associated universal $C^*$-algebra with generators $h_s, s \in \{t_0, \ldots, t_n\}$, such that $h_s \geq 0$ and $\sum_s h_s = 1$. Denote by $I_{\Delta}$ the ideal in $C_{\Delta}$ generated by products of generators containing all the $h_{t_i}, i = 0, \ldots, n$. For each $k$, denote by $I_k$ the ideal in $C_{\Delta}$ generated by all products of generators $h_s$ containing at least $k + 1$ pairwise different generators. We also denote by $I_k^{ab}$ the image of $I_k$ in $C_{\Delta}^{ab}$. We have the following lemma.

Lemma 1.3. [2] Let $\Sigma$ be a locally simplicial complex and $I_n$ be an ideal in $C_{\Sigma}$ defined above. Then isomorphism 
\[ I_k / I_{k+1} \cong \bigoplus_{\Delta} I_{\Delta}, \]
where the sum is taken over all $n$-simplices $\Delta$ in $\Sigma$.

For any vertex $t$ in $\Delta$ there is a natural evaluation map $C_{\Delta} \longrightarrow \mathbb{C}$ mapping the generators $h_t$ to 1 and all the other generators to 0.
Proposition 1.4. (i) The evaluation map $C_\triangle \rightarrow \mathbb{C}$ defined above induces an isomorphism in $K$-theory.
(ii) The surjective map $\mathcal{I}_\triangle \rightarrow \mathcal{I}_\triangle^{ab}$ induces an isomorphism in $K$-theory, where $\mathcal{I}_\triangle^{ab}$ is the abelianization of $\mathcal{I}_\triangle$.

Proof. For (i) it is enough to prove that $C_\triangle$ is homotopy equivalent to $\mathbb{C}$. Consider the $*$-homomorphisms $\alpha : C \rightarrow C_\triangle$, $\lambda \mapsto c_\lambda := \lambda 1$ and $\beta : C_\triangle \rightarrow C$, $h_i \mapsto \frac{1}{n+1} h_i$, $i \in \{0, 1, ..., n\}$. It is clear that $\beta \circ \alpha = id_C$. Define $\varphi_i : C_\triangle \rightarrow C_\triangle$ by $\varphi_i(h_i) = \frac{1}{n+1} + th_i$, $t \in [0, 1]$. It is obvious that $\varphi_1 = id_{C_\triangle}$ and $\varphi_0 = \alpha \circ \beta$. So $\alpha \circ \beta \sim id_{C_\triangle}$. This implies that $C_\triangle$ is homotopy equivalent to $\mathbb{C}$.
Using Lemma 1.3 above, one can use induction on the dimension $n$ of $\triangle$ to prove the claim (ii). For the complete proof we refer to [2].

Remark 1.2. Let $\Delta$ and $\mathcal{I}_\Delta \subset C_\Delta$ as above. Then $K_*(\mathcal{I}_\Delta) \cong K_*(\mathbb{C})$, $* = 0, 1$, if the dimension $n$ of $\Delta$ is even and $K_*(\mathcal{I}_\Delta) \cong K_*(C_0(0, 1))$, $* = 0, 1$, if the dimension $n$ of $\Delta$ is odd.

2. K-THEORY OF NONCOMMUTATIVE CIRCLE

$K$-theory of noncommutative circle was introduced first by [5]. In this article we introduce this algebras as a 1-dimensional simplicial complexes and and it’s skeleton filtration. Basic definitions and facts of $C^*$-algebras, universal $C^*$-algebras and their $K$-theory which we will use in this article can be found in [1], [3], [4], [7] and [8].

Lemma 2.1. $S_1^{nc}/I_1 \cong \mathbb{C}^4$.

Proof. Let $\hat{h}_i$ denote the image of a generator $h_i$ for $S_1^{nc}/I_1$. One has the following relations:

$$\sum_i \hat{h}_i = 1, \quad \hat{h}_i \hat{h}_j = 0, \quad i \neq j.$$

For every $\hat{h}_i$ in $S_1^{nc}/I_1$ we have

$$\hat{h}_i = \hat{h}_i(\sum_i \hat{h}_i) = \hat{h}_i^2.$$ 

Hence $S_1^{nc}/I_1$ is generated by 4 different orthogonal projections and therefore $S_1^{nc}/I_1 \cong \mathbb{C}^4$.

Lemma 2.2. In $S_1^{nc}$, we have an isomorphism

$$I_1/I_2 \cong I_1^{ab}/I_2^{ab}.$$  

specially in $S_1^{nc}/I_1$ we have $I_1/I_2 \cong C_0(0, 1)^4$.

Proof. In $S_1^{ab}$

$$I_1^{ab}/I_2^{ab} \cong \bigoplus_{\sigma} \mathcal{I}_\sigma^{ab}.$$ 

And in $S_1^{nc}$

$$I_1/I_2 \cong \bigoplus_{\sigma} \mathcal{I}_\sigma$$

where the direct sum is taken over all the 1-simplexes $\sigma$ in $\Sigma S_1$. $\mathcal{I}_\sigma$ is the ideal generated by products of generators containing $h_1$ and $h_2$ in the universal $C^*$-algebra $C_\sigma^f$ which is generated by positive elements $h_1$, $h_2$, such that $h_1 + h_2 = 1$. This $C^*$-algebra is commutative. Therefore $\mathcal{I}_\sigma \cong C_0(0, 1)$ and the map $\mathcal{I}_\sigma \rightarrow \mathcal{I}_\sigma^{ab}$ is an isomorphism. In the algebra $S_1^{nc}$, there are four 1-simplexes. So we have $I_1/I_2 \cong C_0(0, 1)^4$.

Lemma 2.3. $C_{\Sigma S_1}$ is commutative.
Proof. An easy computation shows that $C_{\Sigma S^1}/I_2$ is commutative. Since in the algebra $C_{\Sigma S^1}$ the the product of any three different generators is zero, so the ideal $I_2 = 0$. Then $C_{\Sigma S^1}$ is commutative.

Lemma 2.4. $S_{nc}^{nc}/I_2$ is isomorphic to $C_{\Sigma S^1}$.

Proof. Consider

$$\phi : C_{\Sigma S^1} \rightarrow S_{nc}^{nc}/I_2, h_i \mapsto \hat{h}_i, i \in \{0^-, 0^+, 1^-, 1^+\}.$$  

The elements $h_i$ in $C_{\Sigma S^1}$ satisfy the relations of the $\hat{h}_i$ in $S_{nc}^{nc}/I_2$, so $\phi$ is a well defined homomorphism. It is evident that $\phi$ is surjective. It remains to prove that $\phi$ is injective. Let

$$\rho : C_{\Sigma S^1} \rightarrow B(H), h_i \mapsto g_i$$

be a unital representation. So in $B(H)$, we have

$$\sum g_i = \sum \rho(h_i) = \rho(\sum h_i) = \rho(1) = 1$$

and all $g_i$ commute since $C_{\Sigma S^1}$ is commutative. Now, define

$$\pi : S_{nc}^{nc} \rightarrow B(H), \pi(\hat{h}_i) = g_i.$$  

Then $\pi$ annihilates $I_2$ and therefore factors as

$$S_{nc}^{nc} \rightarrow S_{nc}^{nc}/I_2 \xrightarrow{\pi'} B(H)$$

where $\pi'$ is a well defined homomorphism such that $\rho = \pi' \circ \phi$.

Proposition 2.5. $S_{nc}^{nc}/I_2 \cong S_{ab}^{ab}$.

Proof. By 2.4, $S_{nc}^{nc}/I_2$ is isomorphic to the commutative algebra $C_{\Sigma S^1}$. Thus $S_{nc}^{nc}/I_2$ is an abelian $C^*$-algebra. Consider the following commutative diagram

$$\begin{array}{ccccccccc}
0 & \rightarrow & I_1/I_2 & \rightarrow & S_{nc}^{nc}/I_2 & \rightarrow & S_{nc}^{nc}/I_1 & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \rightarrow & I_{1}^{ab}/I_{2}^{ab} & \rightarrow & S_{1}^{ab}/I_{2}^{ab} & \rightarrow & S_{1}^{ab}/I_{1}^{ab} & \rightarrow & 0.
\end{array}$$

Since

$$S_{nc}^{nc}/I_1 \cong S_{1}^{ab}/I_1 \cong \mathbb{C}^d$$

from Lemma 2.1 and

$$I_1/I_2 \cong I_{1}^{ab}/I_{2}^{ab}$$

from lemma 2.2 By five-lemma, we get

$$S_{nc}^{nc}/I_2 \cong S_{1}^{ab}/I_{2}^{ab}$$

In $S_{1}^{ab}$, we have $I_{2}^{ab} = 0$. So

$$S_{nc}^{nc}/I_2 \cong S_{1}^{ab} = C(S^1).$$

Remark 2.1. We have that

$$C(|\Sigma S^1|) \cong C(S^1),$$

since $|\Sigma S^1|$ and $S^1$ are homeomorphic spaces.

We now consider the simplicial flag complex $\Lambda$ with 3 vertices $\{1, 2, 3\}$ such that $\{1, 3\} \notin \Lambda$. 


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Lemma 2.6. The universal $C^*$-algebra $C^f_{\Lambda}$ generated by positive generators $h_1, h_2, h_3$ with sum equal to one and $h_1h_3 = 0$ is homotopy equivalent to $\mathbb{C}$.

Proof. Let $\alpha : C^f_{\Lambda} \longrightarrow \mathbb{C}$ be the homomorphism which sends $h_2$ to 1 and $h_1, h_3$ to 0. And let $\beta : \mathbb{C} \longrightarrow C^f_{\Lambda}$ be the natural homomorphism which sends 1 in $\mathbb{C}$ to the identity element in $C^f_{\Lambda}$.

It’s clear that $\alpha \circ \beta = id_{\mathbb{C}}$. Define

$$\varphi_t : C^f_{\Lambda} \longrightarrow C^f_{\Lambda},$$

by mapping $h_2$ to $h_2 + (1-t)(h_1 + h_3)$ and $h_i$ to $th_i$ for $i = 1, 3$. The $\varphi_t(h_i)$ satisfy the following relations:

(i) $\varphi_t(h_i) \geq 0 \forall i \in \{1, 2, 3\}$,

(ii) $\varphi_t(h_1) + \varphi_t(h_2) + \varphi_t(h_3) = th_1 + (h_2 + (1-t)(h_1 + h_3)) + th_3 = h_1 + h_2 + h_3 = 1$.

(iii) $\varphi_t(h_1)\varphi_t(h_3) = th_1th_3 = t^2h_1h_3 = 0$.

Since the elements $\varphi_t(h_i)$ satisfy the relations of the $h_i$ in $C^f_{\Lambda}$, $\varphi_t$ is well defined.

It is obvious that $\varphi_1 = id_{C^f_{\Lambda}}$ and $\varphi_0 = \beta \circ \alpha$. This means that $\beta \circ \alpha$ is homotopic to $Id_{C^f_{\Lambda}}$. Hence it follows that $C^f_{\Lambda}$ is homotopy equivalent to $\mathbb{C}$. \qed

Lemma 2.7. In the previous lemma, let $\mathcal{I}_{\Lambda}$ be the ideal in $C^f_{\Lambda}$ generated by the products containing all generators $h_1, h_2, h_3$. Then $\mathcal{I}_{\Lambda}$ is homotopy equivalent to zero.

Proof. We have from the previous lemma that

$$\varphi_t : C^f_{\Lambda} \longrightarrow C^f_{\Lambda},$$

is well defined.

We show that $\varphi_t$ maps $\mathcal{I}_{\Lambda}$ to $\mathcal{I}_{\Lambda}$ and therefore induces by restriction a homomorphism

$$\varphi_t|_{\mathcal{I}_{\Lambda}} := \hat{\varphi}_t : \mathcal{I}_{\Lambda} \longrightarrow \mathcal{I}_{\Lambda}.$$ 

Let $x = \ldots h_1 h_j h_3 \ldots$ be a typical element in $\mathcal{I}_{\Lambda}$. We have

$$\hat{\varphi}_t(h_1 h_j h_3) = \varphi_t(h_1) \varphi_t(h_j) \varphi_t(h_3)$$

$$= th_1(h_2 + (1-t)(h_1 + h_3)^k)th_3 = h_1 P(h_2)h_3$$

where $P$ is polynomial without constant term. So the product is in $\mathcal{I}_{\Lambda}$. Note that we used in the equations above that $h_1h_3 = 0$.

It is clear that $\hat{\varphi}_0 = 0$ and $\hat{\varphi}_1 = id_{\mathcal{I}_{\Lambda}}$.

This yields that $\mathcal{I}_{\Lambda}$ is homotopy equivalent to zero. \qed

Lemma 2.8. For the skeleton filtration $(I_k)$ in $S^{nc}_1$, $I_2/I_3$ has trivial $K$-theory.

Proof. Consider the skeleton filtration

$$S^{nc}_1 := I_0 \supset I_1 \supset I_2 \supset I_3.$$ 

By Lemma 1.1 we have

$$I_2/I_3 \cong \bigoplus_{\Lambda_{i}} \mathcal{I}_{\Lambda_{i}},$$

where $\Lambda_{i}$ is the subcomplex of $\Sigma_{S^1}$ generated by $\{0^+, 0^-, 1^+, 1^-\} \setminus \{i\}$, and $\mathcal{I}_{\Lambda_{i}}$ is the ideal generated by products containing all generators $h_j$, $j \in V_{\Sigma_{S^1}} \setminus \{i\}$.

There are four orthogonal ideals of this form. The orthogonality is clear, since e.g. if $x \in \mathcal{I}_{\Lambda_{i}^+}$ and $y \in \mathcal{I}_{\Lambda_{i}^-}$, the product

$$xy = (\ldots h_1 h_0 h_1 \ldots)(\ldots h_1 h_0 h_1 \ldots)$$

contains four different generators, so it is equal to zero in $I_2/I_3$.

Using Lemma 2.7 we get that $\mathcal{I}_{\Lambda_{i}}$ is homotopic to zero. This implies that $K_*(I_2/I_3) = 0$. \qed
Proposition 2.9. In $S^{nc}_1$ we have $K_*(I_2) = K_*(I_3), * = 0, 1$.

Proof. We construct the short exact sequence

$$0 \rightarrow I_3 \rightarrow I_2 \rightarrow I_2/I_3 \rightarrow 0.$$  

Apply the six-term exact sequence and use the lemma above. We get two isomorphisms in $K$-theory $K_*(I_2) \cong K_*(I_3), * = 0, 1$.

Proposition 2.10. We have

$$K_*(S^{nc}_1/I_3) \cong K_*(S^{nc}_1/I_2) \cong K_*(C(S^1)).$$

Proof. Consider the short exact sequence

$$0 \rightarrow I_2/I_3 \rightarrow S^{nc}_1/I_3 \rightarrow S^{nc}_1/I_2 \rightarrow 0.$$  

Applying the six-term exact sequence, we get

$$K_0(I_2/I_3) \rightarrow K_0(S^{nc}_1/I_3) \rightarrow K_0(S^{nc}_1/I_2) \rightarrow K_1(S^{nc}_1/I_2) \leftarrow K_1(S^{nc}_1/I_3) \leftarrow K_1(I_2/I_3).$$

From Lemma 2.8 we have $K_*(I_2/I_3) = 0$, so that the above six-term exact sequence reduces to the following two isomorphisms

$$K_0(S^{nc}_1/I_3) \cong K_0(S^{nc}_1/I_2)$$

and

$$K_1(S^{nc}_1/I_3) \cong K_1(S^{nc}_1/I_2).$$

Note that by Lemmas 2.5 and 2.6, the $C^*$-algebras $S^{nc}_1/I_2$ and $C(S^1)$ have the same $K$-theory. This proves the proposition.

Proposition 2.11. In the algebra $S^{nc}_1$, we have $K_0(I_2) = K_0(I_3) = \mathbb{Z}$ and $K_1(I_2) = K_1(I_3) = 0$.

Proof. $I_2$ is a closed two-sided ideal in $S^{nc}_1$. We have the following short exact sequence

$$0 \rightarrow I_2 \rightarrow S^{nc}_1 \xrightarrow{i} S^{nc}_1/I_2 \rightarrow 0.$$  

During the rest of this section, denote $K_*(i)$ by $i_*$ and $K_*(\pi)$ by $\pi_*$ for $* = 0, 1$. From the above exact sequence we obtain the following six-term exact sequence.

$$K_0(I_2) \rightarrow K_0(S^{nc}_1) \xrightarrow{\pi_0} K_0(S^{nc}_1/I_2) \rightarrow K_1(S^{nc}_1/I_2) \leftarrow K_1(S^{nc}_1) \leftarrow K_1(I_2).$$

We have from Theorem 1.2

$$K_*(S^{nc}_1) \cong K_*(C),$$

which is generated by $[1_{S^{nc}_1}]$, where $1_{S^{nc}_1}$ denotes the identity element in $S^{nc}_1$. And from the above lemma we have

$$K_*(S^{nc}_1/I_2) \cong K_*(C(S^1)).$$

It’s well known that $K_*(C(S^1)) \cong \mathbb{Z}$, for $* = 0, 1$ so, the above six-term exact sequence reads as

$$K_0(I_2) \rightarrow \mathbb{Z} \xrightarrow{\pi_0} \mathbb{Z} \rightarrow K_1(I_2).$$
With respect to the isomorphism \( K_0(S_1^{nc}/I_2) \cong K_0(C(S^1)) \), the image \( \pi_0([1_{S_1^{nc}}]) \) of the generator of \( K_0(S_1^{nc}) \) corresponds to the generator \( [1_{C(S^1)}] \) of \( K_0(C(S^1)) \).

So \( \pi_0 \) is bijective. Then \( i_0 \) is zero, and we have \( K_0(I_2) = \mathbb{Z} \) and \( K_1(I_2) = 0 \). By proposition 2.9, we have also \( K_0(I_3) = \mathbb{Z} \) and \( K_1(I_3) = 0 \).

**Proposition 2.12.** Consider the skeleton filtration

\[ S_1^{nc} = I_0 \supset I_1 \supset I_2 \supset I_3. \]

The short exact sequence

\[ 0 \longrightarrow I_k \overset{i}{\longrightarrow} I_{k-1} \overset{\pi}{\longrightarrow} I_{k-1}/I_k \longrightarrow 0 \]

induces \( i_* : K_*(I_k) \longrightarrow K_*(I_{k-1}) \) which is zero for \( 1 \leq k \leq 2 \), and \( * = 0, 1 \).

**Proof.** For \( k = 1 \), we have the following six-term exact sequence

\[
\begin{array}{c}
K_0(I_1) \overset{i_0}{\longrightarrow} K_0(S_1^{nc}) \overset{\pi_0}{\longrightarrow} K_0(S_1^{nc}/I_1) \\
\downarrow \quad \downarrow \\
K_1(S_1^{nc}/I_1) \overset{i_1}{\longrightarrow} K_1(S_1^{nc}) \overset{\pi_1}{\longrightarrow} K_1(I_1).
\end{array}
\]

From Theorem 2.12, \( K_*(S_1^{nc}) \cong K_*(C(S^1)) \) and by Lemma 2.1, \( K_0(S_1^{nc}/I_1) \cong \mathbb{Z}^4 \) and \( K_1(S_1^{nc}/I_1) = 0 \). So there is an embedding \( \mathbb{Z} \overset{\pi_0}{\longrightarrow} K_0(S_1^{nc}/I_1) \), therefore \( i_0 = 0 \). It is already \( i_1 = 0 \). Moreover, it is also clear that \( K_0(I_1) = 0 \). For \( k = 2 \), we get the six-term exact sequence

\[
\begin{array}{c}
K_0(I_2) \overset{i_0}{\longrightarrow} K_0(I_1) \overset{\pi_0}{\longrightarrow} K_0(I_1/I_2) \\
\downarrow \quad \downarrow \\
K_1(I_1/I_2) \overset{i_1}{\longrightarrow} K_1(I_1) \overset{\pi_1}{\longrightarrow} K_1(I_2).
\end{array}
\]

From above \( K_0(I_1) = 0 \), and from proposition 2.11 \( K_1(I_2) = 0 \), so \( i_* = 0 \).

For \( k = 3 \), proposition 2.9 gives a counterexample, since \( i_* \) is an isomorphism between \( K_*(I_2) \) and \( K_*(I_3) \), and therefore \( i_* \neq 0 \) for \( k = 3 \).

**References**


